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Fast Restoration of Finite Objects Degraded by Finite PSF*

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The problem of restoring a class of continuous, finite random objects degraded by finite space invariant PSF and additive noise is considered. For infinite images this problem is solved easily via Fourier domain Wiener filtering. For finite intervals, this solution is no longer valid. Here we show that a modification of the image data by the boundary observations gives a solution of the Wiener filtering problem in such a way that the restored object is obtained exactly in terms of a Fourier series expansion which can be implemented in practice via a fast Fourier transform algorithm. Examples of two-dimensional images are given.

This paper considers the problem of Wiener filtering of images of finite random objects degraded by a finite space invariant point spread function (PSF) and additive white noise. Let $u(x)$ represent a sample function of a class of wide sense stationary random objects, i.e.,

$$E\{u(x)\} = \mu, \tag{1}$$

$$E\{u(x)\, u(x')\} = r(x - x'). \tag{2}$$

Further assume that $u(x)$ is defined on a finite interval $[-L, L]$, so that Eqs. (1) and (2) are valid only for $-L \leqslant x, x' \leqslant L$. Without loss of generality, we will assume that $\mu = 0$ and we will specialize (2) for the class of objects whose autocorrelation function is given by

$$r(x - x') = e^{-\alpha|x-x'|}, \qquad -L \leqslant x,\ x' \leqslant L. \tag{3}$$

If $h(x)$ is a symmetric, spatially invariant PSF whose range does not exceed $2L$, i.e.,

$$h(x) = h(-x), \qquad -L \leqslant -L_1 \leqslant x \leqslant L_1 \leqslant L, \tag{4}$$

$$h(x) = 0, \qquad |x| > L_1, \tag{5}$$

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then the observed image is given by

$$y(x) = \int_{-\infty}^{\infty} h(x - x')\, u(x')\, dx' + w(x), \tag{6}$$

where $w(x)$ is zero mean white Gaussian noise with variance σ_n^2. Since $u(x)$ is defined in a finite interval, Eq. (6) can be written as

$$y(x) = \int_{-L}^{L} h(x - x')\, u(x')\, dx' + w(x), \qquad -2L \leqslant x \leqslant 2L, \tag{7a}$$

$$= w(x), \qquad |\, x \,| > 2L. \tag{7b}$$

From (7a) it is noted that since the range of $h(x)$ can be as much as $[-L, L]$, $y(x)$ is defined over $[-2L, 2L]$. In Eq. (7b), $w(x)$ may be interpreted as the object background and is assumed to be a white noise random field uncorrelated with the objects, i.e.,

$$E\{u(x)\, w(x')\} = 0 \qquad \forall x, x'. \tag{8}$$

The Complexities of Wiener Filtering

The filtering problem is to find the best mean square estimate of $u(x)$ given the observations $y(x)$ over the finite interval $[-2L, 2L]$. The solution of this problem is quite straightforward via Fourier theory when $u(x)$ is defined as a stationary object over the infinite interval $(-\infty, \infty)$. In fact, for $L = \infty$, one solves the Wiener filtering problem associated with (6) which gives the filtered estimate $u^*(x)$ as the Fourier inverse of $U^*(\omega)$ given by [4]

$$U^*(\omega) = \frac{\Phi_u(\omega)\, H(\omega)}{|\, H(\omega)|^2\, \Phi_u(\omega) + \sigma_n^2}, \tag{9}$$

where $Y(\omega)$, $H(\omega)$ are the Fourier transforms of $y(x)$, $h(x)$, respectively, and $\Phi_u(\omega)$ is the power spectrum of $u(x)$. The Wiener filter equation (9) is obtained by first writing the Fourier transform of (6) as

$$Y(\omega) = H(\omega)\, U(\omega) + W(\omega) \tag{10}$$

and noting that $U(\omega)$ and $W(\omega)$ are uncorrelated and that each is an uncorrelated process in the frequency domain, i.e.,

$$E\{U(\omega)\, U(\omega')\} = \Phi_u(\omega)\, \delta(\omega - \omega'), \tag{11}$$

$$E\{W(\omega)\, W(\omega')\} = \sigma_n^2 \delta(\omega - \omega'), \tag{12}$$

$$E\{U(\omega)\, W(\omega')\} = 0. \tag{13}$$

Let $\Phi_e(\omega)$ denote the power spectral density of the error, $u(x) - u^*(x)$. Assuming a

Gaussian distribution of the random object $u(x)$, we can treat $Y(\omega)$ as independent for each different ω and write $U^*(\omega) = F(\omega)\, Y(\omega)$ for each ω, as a scalar filter equation. The $F(\omega)$ is determined such that $\int \Phi_e \, d\omega$ is minimized. Equation (9) is then found to be the resulting solution. However, when $u(x)$ is defined on a finite interval $[-L, L]$, Eq. (9) no longer gives the optimum estimate. This is because Eq. (11) no longer holds and the $Y(\omega)$ are correlated for different ω values.

Over a finite interval it is possible to find an expansion for $u(x)$ in the form

$$u(x) = \sum_{i=0}^{\infty} u_i \phi_i(x), \qquad -L \leqslant x \leqslant L, \tag{14}$$

such that $\{\phi_i\}$ is a set of complete orthonormal (CON) functions over $[-L, L]$ and the coefficients u_i are uncorrelated, viz.,

$$E\{u_i u_j\} = \lambda_i \delta_{ij}, \tag{15}$$

where

$$u_i = \int_{-L}^{L} u(x)\, \phi_i(x)\, dx. \tag{16}$$

This expansion is called the Karhunen–Loeve expansion [1], and the $\{\phi_i\}$ are given as the solutions of the eigenvalue problem.

$$\int_{-L}^{L} r(x - x')\, \phi_i(x')\, dx' = \lambda_i \phi_i(x). \tag{17}$$

Since $\{\phi_i\}$ are complete and orthonormal we can expand $h(x)$, in terms of ϕ_i over $[-L, L]$, to give

$$h(x) = \sum_{i=0}^{\infty} h_i \phi_i(x), \qquad -L \leqslant x \leqslant L, \tag{18}$$

$$h_i = \int_{-L}^{L} h(x)\, \phi_i(x)\, dx. \tag{19}$$

Unfortunately $y(x)$ is defined over $[-2L, 2L]$, and since $\phi_i(x)$ are not necessarily complete and orthogonal over $[-2L, 2L]$, we cannot expand $y(x)$ in terms of $\phi_i(x)$. Even if we try to expand $y(x)$ over $[-L, L]$ we have difficulty because the expansion of $h(x)$ in (18) is valid only when its argument x is in the $[-L, L]$ interval. However, in (7a) the argument of $h(x - x')$ has a range of $[-2L, 2L]$ when x and x' are in $[-L, L]$. Hence a possible solution of (7a) is to find the CON functions ϕ_i as solution of the following equations.

$$\int_{-L}^{L} r(x - x')\, \phi_i(x')\, dx' = \lambda \phi_i(x), \qquad -L \leqslant x \leqslant L, \tag{20}$$

and

$$\int_{-2L}^{2L} h(x - x')\, \phi_i(x)\, dx' = \gamma_i \phi_i(x). \tag{21}$$

This is called a doubly orthogonal expansion, since the functions ϕ_i must simultaneously solve the two different eigenvalue problems. It is obvious that if $L = \infty$, $\phi(x) = e^{\pm j\omega x}$ are two solutions and λ and γ are the Fourier transforms of $r(x)$ and $h(x)$, respectively.

In general, the solutions of such equations (if they exist) are not available in closed form and have to be computed numerically. Even in the special case when $r(x - x')$ is given by (3) and $h(x)$ is given in (4) and (5), a closed form solution for $\phi(x)$ is unknown. And, even if $h(x - x') = r(x - x')$ the simultaneous solution of (20), (21) is not the same as that of (20) alone, because the interval of x in (21) is $(-2L, 2L)$ and is different from that in (20). For example, it is known that when r satisfies Eq. (3), the solution of (20) alone is given by [2]

$$\begin{aligned} \phi_i(x) &= \frac{1}{L^{1/2}(1 + (\sin 2b_iL)/2b_iL)} \cos b_i x, \qquad i = 1, 3, 5,..., \\ &= \frac{1}{L^{1/2}(1 - (\sin 2b_iL)/2b_iL)} \sin b_i x, \qquad i = 2, 4, 6,..., \end{aligned} \tag{22}$$

for $-L \leqslant x \leqslant L$ and

$$\lambda_i = \frac{2\alpha}{\alpha^2 + b_i^2}, \qquad i = 1, 2, 3,..., \tag{23}$$

where b_i are solutions of the transcendental equation

$$(\tan b_iL + b_i/\alpha)(\tan b_iL - \alpha/b_i) = 0. \tag{24}$$

It can be checked that the CON functions $\{\phi_i(x)\}$ are *not solutions* of Eq. (21) even when $h(x - x') = r(x - x')$, i.e., of the integral equation

$$\int_{-2L}^{2L} e^{-\alpha|x-x'|} \phi(x')\, dx' = \gamma\phi(x), \qquad -2L \leqslant x \leqslant 2L. \tag{25}$$

Further, it is to be noted that if $h(x - x') = \delta(x - x')$, solutions of (20) are also solutions of (21), which is to say that if the object is degraded by noise alone, then the Karhunen–Loeve expansion of the object can be used to expand the image as well as the noise $w(x)$. Thus (if $h(x - x') = \delta(x - x')$),

$$z(x) = \sum_i z_i \phi_i(x), \qquad z = u, w, y, \tag{26}$$

$$z_i = \int_{-L}^{L} z(x)\, \phi_i(x)\, dx, \qquad z_i = u_i, w_i, y_i, \tag{27}$$

and

$$y_i = u_i + w_i \,. \tag{28}$$

Since ϕ_i are solutions of (20) and $w(x)$ is white noise, it is easy to check that both $\{u_i\}$ and $\{w_i\}$ are uncorrelated sequences, and the Wiener filter is given by

$$u_i^* = \frac{\lambda_i}{\lambda_i + \sigma_n^2}\, y_i \tag{29}$$

and the best mean square estimate of $u(x)$ is reconstructed as

$$u^*(x) = \sum_{i=0}^{\infty} u_i^* \phi_i(x). \tag{30}$$

It is noted that even for this simple problem (PSF = delta function) computationally the transcendental equation of (24) is to be solved and the computations in (27) and (30) each require NM multiplications if N discrete steps are chosen in evaluating the integral (27) and M terms are chosen in the sines summation of (30). The computational complexity of solving the original filtering problem via numerical solutions of (20) and (21) becomes even greater.

In this paper we show that if there is no uncertainty at the boundary points of the object (i.e., $u(-L)$ and $u(L)$ are given), the equations for the observations $y(x)$ and for the Karhunen–Loeve expansion eigenvalue problem can be modified by this boundary information in such a way that the Wiener filtering problem is solved by a set of harmonic sinusoidal functions leading to a fast computational implementation of the filter equations.

Random Object Statistics with Known Boundaries

Suppose the two boundary values of the object, viz., $u(-L)$, $u(L)$, are given. Then we define a new function $\hat{u}(x)$ as

$$\hat{u}(x) = u(x) - a(x)\, u(-L) - b(x)\, u(L), \tag{31}$$

where $a(x)$ and $b(x)$ are weights such that the expected square value $E\{(\hat{u}(x))^2\}$ for any x in $[-L, L]$ is minimized. The implication here is that $a(x)$, $b(x)$ represent the weight of the boundary information in the random function $u(x)$. The quantity

$$a(x)\, u(-L) + b(x)\, u(L) \triangleq u_b(x) \tag{32}$$

may be interpreted as the "boundary information" or "Boundary Response Information" in $u(x)$, and Eq. (31) is rewritten as

$$u(x) = \hat{u}(x) + u_b(x). \tag{33}$$

Therefore, the random process $u(x)$ is decomposed as a sum of two random processes $\hat{u}(x)$ and $u_b(x)$, where $u_b(x)$ contains only boundary information and is also the best mean square estimate of $u(x)$ if the boundary information alone is given. The weights $a(x)$ and $b(x)$ are obtained as

$$a(x) = \frac{\sinh \alpha(L - x)}{\sinh 2\alpha L}, \tag{34}$$

$$b(x) = \frac{\sinh \alpha(L + x)}{\sinh 2\alpha L}. \tag{35}$$

The derivation of $a(x)$ and $b(x)$ is not important for the restoration problem considered here and is, therefore, omitted for the sake of brevity. Equation (31) could just be taken as a definition of a modified object obtained from the original object and its boundary values. The mean and covariance functions of the modified object are obtained directly from (31), (34), and (35) as

$$\begin{aligned} E\{\hat{u}(x)\} &= 0, \\ E\{\hat{u}(x)\, \hat{u}(x')\} &= \hat{r}(x, x') \\ &= e^{-\alpha|x-x'|} - e^{-\alpha L}\left[\frac{\cosh \alpha x}{\cosh \alpha L} \cosh \alpha x' + \frac{\sinh \alpha x}{\sinh \alpha L} \sinh \alpha x'\right]. \end{aligned} \tag{36}$$

It is observed that the process $u(x)$ is no longer stationary. Solution of the integral equation (20) with this modified kernel proceeds in a manner identical to that for the unmodified kernel [2]. The solutions are now harmonic sinusoids that form a CON set of basis functions and are given in closed form as

$$\begin{aligned} \hat{\phi}_i(x) &= \left(\frac{1}{L}\right)^{1/2} \cos \frac{i\pi}{2L} x, \qquad i = 1, 3, 5,..., \\ &= \left(\frac{1}{L}\right)^{1/2} \sin \frac{i\pi}{2L} x, \qquad i = 2, 4, 6,..., \end{aligned} \qquad -L \leqslant x \leqslant L, \tag{37}$$

$$\hat{\lambda}_i = \frac{2\alpha}{\alpha^2 + (i\pi/2L)^2}, \qquad i = 1, 2, 3,... . \tag{38}$$

Note here that the eigenfunctions $\hat{\phi}_i(x)$, in addition to being harmonics, depend only on L and are independent of α. The fact that $\{\phi_i(x)\}$ are a CON set of eigenfunctions of $\hat{r}(x, x')$ can be verified by converting the integral equation

$$\int_{-L}^{L} r(x, x')\, \hat{\phi}_i(x')\, dx' = \hat{\lambda}_i \hat{\phi}_i(x) \tag{39}$$

into a second-order ordinary differential equation by a method similar to that of VanTrees [2], and (37) and (38) are solutions of that equation. These functions are

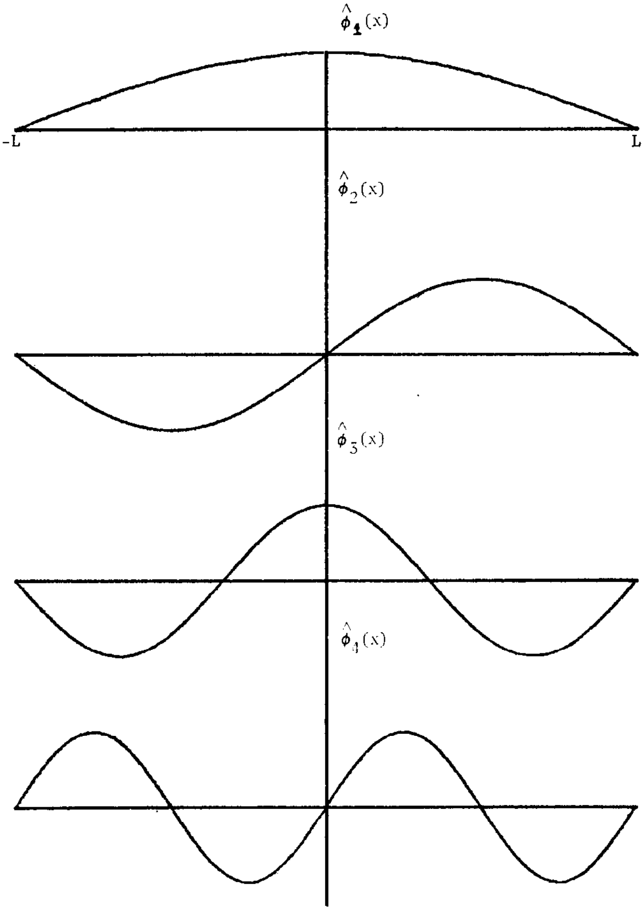

FIG. 1. Eigenfunctions $\hat{\phi}_i(x)$.

equivalent to those obtained by Jain [3, 5] for the discrete case except for a shift in origin. (Fig. 1)

The process $\hat{u}(x)$ can now be expanded by its Karhunen–Loeve expansion

$$\hat{u}(x) = \sum_i \hat{u}_i \hat{\phi}_i(x),$$
$$\hat{u}_i = \int_{-L}^{L} \hat{u}(x)\, \hat{\phi}_i(x)\, dx. \tag{40}$$

Using (36) and (37) in the latter equation, one may easily see that

$$\begin{aligned} E\{\hat{u}_i \hat{u}_j\} &= \hat{\lambda}_i, \qquad i = j, \\ &= 0, \qquad i \neq j, \end{aligned} \tag{41}$$

which, of course, is also the property of a Karhunen–Loeve expansion.

The functions $\hat{\phi}_i(x)$ are much simpler to generate, and the computations involved in (39) and (40) are greatly reduced because they are related to the Fourier series and a fast Fourier transform [7] algorithm can be utilized if these computations are performed on a digital computer. If (39) and (40) are to be implemented by analog signal generators and correlators, the implementation is greatly simplified, since only harmonic sinusoids, independent of the object statistical parameter α, need be generated.

From (37) it is seen that the functions $\hat{\phi}_i(x)$ are *not* the Fourier series sinusoids over the interval $[-L, L]$. In fact, the Fourier series is given in terms of the pair of functions

$$\psi_i(x) = \sin \frac{i\,2\pi x}{2l}, \qquad \cos \frac{i2\pi x}{2l}, \qquad i = 0, 1, 2, \ldots, \qquad -l \leqslant x \leqslant l,$$

over any interval $(-l, l)$. Thus, for $l = L$, $\psi_i(x)$ and $\hat{\phi}_i(x)$ are different. However, if $l = 2L$, then the functions $\hat{\phi}_i(x)$ extended in the region $[-2L, 2L)$ form a subset of the Fourier series functions $\psi_i(x)$. This means we can use $\hat{\phi}_i(x)$ over $[-L, L]$ to expand the modified object and use the Fourier series to expand the PSF $h(x)$ as well as the modified observations $\hat{y}(x)$ over $[-2L, 2L]$. It is then seen that only a partial set of the Fourier expansion of $\hat{y}(x)$ is sufficient for the restoration problem.

Optimum Restoring Filter for Symmetric Space Invariant PSF

Suppose now that a sample function of $u(x)$ is blurred by a known symmetric space-limited PSF $h(x)$ and corrupted by zero mean white noise $w(x)$ with variance σ_n^2. The output $y(x)$ is observed.

$$y(x) = \int_{-\infty}^{\infty} u(\beta)\, h(x - \beta)\, d\beta + w(x), \qquad -\infty \leqslant x \leqslant \infty. \tag{42}$$

With the boundary values $u(-L)$, $u(L)$ given, modify this observation as

$$\begin{aligned}\hat{y}(x) &= y(x) - \frac{1}{\sinh 2\alpha L}\left[\int_{-\infty}^{\infty} h(x-\beta)[\sinh \alpha(L-\beta)\, u(-L) \right.\\ &\qquad \left. + \sinh \alpha(L+\beta)\, u(L)]\, d\beta\right]\\ &= \int_{-\infty}^{\infty} \hat{u}(\beta)\, h(x-\beta)\, d\beta + w(x)\\ &= \begin{cases} \int_{-L}^{L} \hat{u}(\beta)\, h(x-\beta)\, d\beta + w(x), & |x| \leqslant 2L,\\ w(x), & |x| > 2L,\end{cases}\end{aligned}$$

where $\hat{u}(x)$ is the modified random process of (31). Since the PSF, $h(x)$, is assumed to be narrower than $2L$, i.e., $h(x) = 0$, $|x| > L$. It is expanded in a Fourier series over the interval $[-2L, 2L]$, since $\hat{y}(x)$ exists over $[-2L, 2L]$.

$$h(x) = \frac{1}{2L} \sum_{i=0}^{\infty} h_i \cos \frac{i\pi}{2L} x, \qquad -3L \leqslant x \leqslant 3L,$$
$$h_i = \int_{-2L}^{2L} h(x) \cos \frac{i\pi}{2L} x \, dx = \int_{-L}^{L} h(x) \cos \frac{i\pi}{2L} x \, dx. \tag{44}$$

Note here that the series expansion of $h(x)$ is valid over the interval $[-3L, 3L]$ even though the function was expanded over $[-2L, 2L]$. Equation (44) describes a function which is periodic with period $4L$ and which is 0 in the intervals $[-2L, -L]$ and $[L, 2L]$. Therefore, it must be 0 over $[-3L, -2L]$ and $[2L, 3L]$. This allows us to substitute (44) for $h(x - \beta)$ in (43), where the interval of $x - \beta$ is $[-3L, 3L]$.

$$\begin{aligned} \hat{y}(x) &= \frac{1}{2L} \sum_{i=0}^{\infty} h_i \int_{-L}^{L} \hat{u}(\beta) \cos \frac{i\pi}{2L} (x - \beta) \, d\beta + w(x) \\ &= \frac{1}{2L} \sum_{i=0}^{\infty} h_i \cos \frac{i\pi}{2L} x \int_{-L}^{L} \hat{u}(\beta) \cos \frac{i\pi}{2L} \beta \, d\beta \\ &\quad + \frac{1}{2L} \sum_{i=0}^{\infty} h_i \sin \frac{i\pi}{2L} x \int_{-L}^{L} \hat{u}(\beta) \sin \frac{i\pi}{2L} \beta \, d\beta + w(x), \qquad -2L \leqslant x \leqslant 2L. \end{aligned} \tag{45}$$

The process can now be decomposed by correlating both sides of (45) with harmonic sinusoids, which gives

$$\begin{aligned} \hat{y}_i &= \left(\frac{1}{L}\right)^{1/2} \int_{-2L}^{2L} \hat{y}(x) \cos \frac{i\pi}{2L} x \, dx = h_i \hat{u}_i + w_i, \qquad i = 1, 3, 5, \ldots, \\ &= \left(\frac{1}{L}\right)^{1/2} \int_{-2L}^{2L} \hat{y}(x) \sin \frac{i\pi}{2L} x \, dx = h_i \hat{u}_i + w_i, \qquad i = 2, 4, 6, \ldots, \end{aligned} \tag{46}$$

where the $\hat{u}_i$ are defined in (40). The w_i are given by

$$\begin{aligned} w_i &= \left(\frac{1}{L}\right)^{1/2} \int_{-2L}^{2L} w(x) \cos \frac{i\pi}{2L} x \, dx, \qquad i = 1, 3, 5, \ldots, \\ &= \left(\frac{1}{L}\right)^{1/2} \int_{-2L}^{2L} w(x) \sin \frac{i\pi}{2L} x \, dx, \qquad i = 2, 4, 6, \ldots, \end{aligned} \tag{47}$$

FIG. 3. Reconstruction of fig. 2.

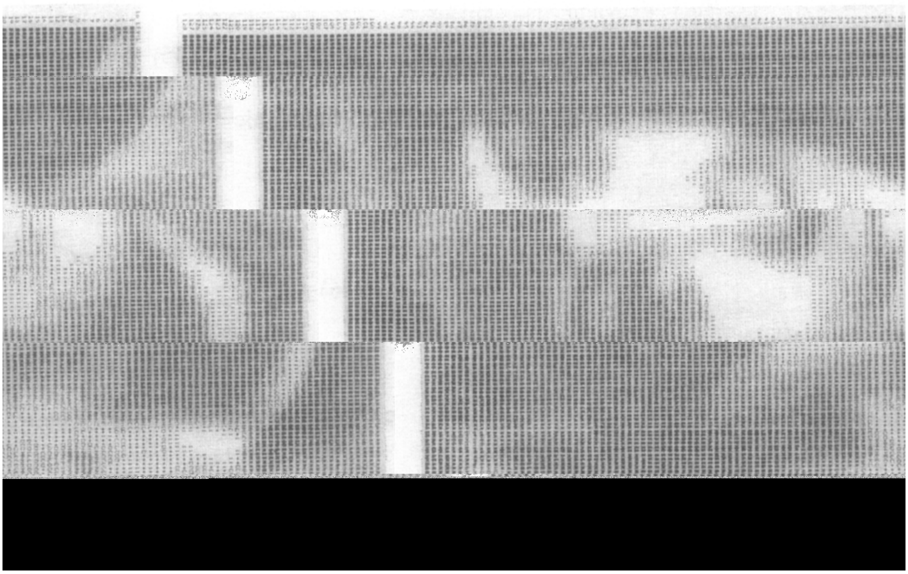

FIG. 2. Blurred image with no noise.

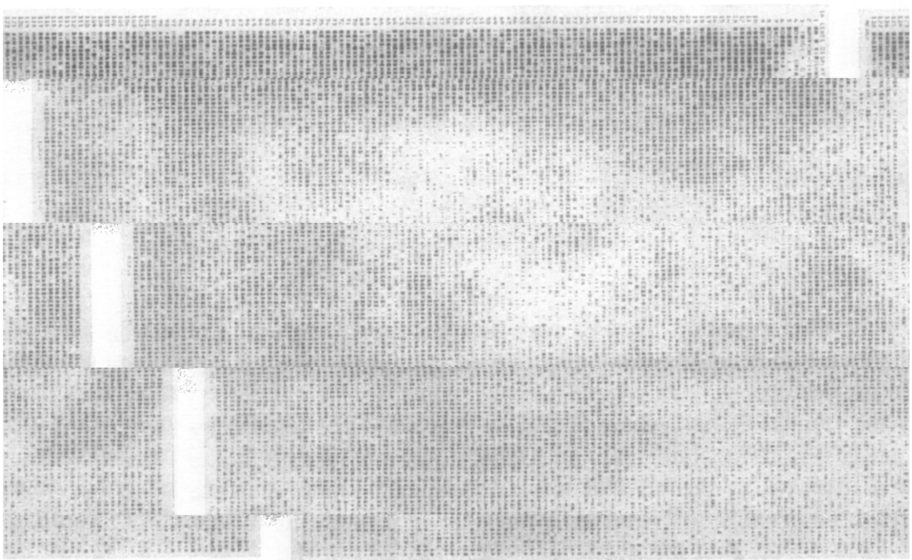

FIG. 4. Noisy blurred image. SNR ~ 6 db.

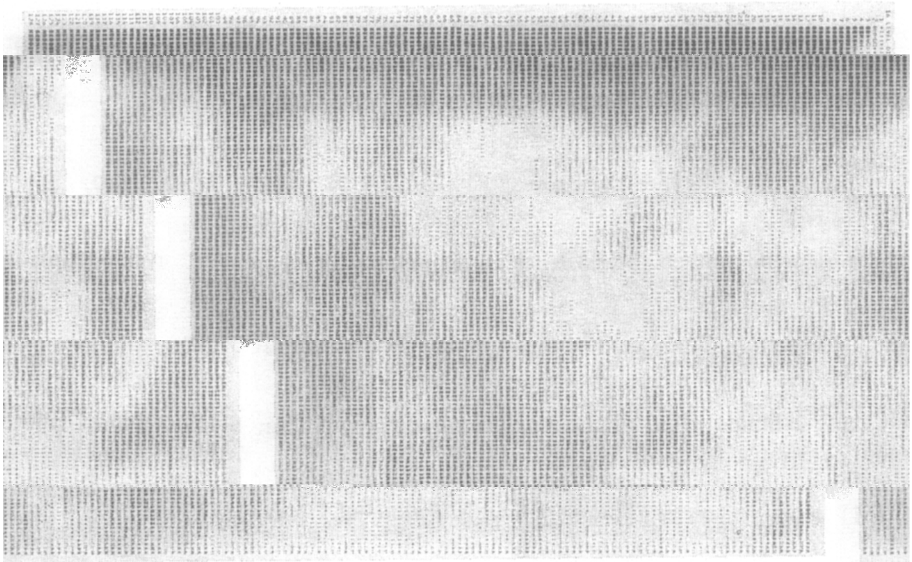

FIG. 5. Reconstruction of fig. 4.

with

$$E\{w_i\} = 0, \qquad E\{w_i w_j\} = \sigma_n^2 \delta_{ij} . \tag{48}$$

Since h_i are deterministic and both $\{\hat{u}_i\}$ and $\{w_i\}$ are uncorrelated sequences, we can derive a Wiener filter for each $\hat{y}_i$ independently.

The Wiener filter minimum mean square estimates of $\hat{u}_i$ are given by

$$\hat{u}_i^* = \frac{h_i \lambda i}{h_i^2 \lambda_i + \sigma_n^2} \hat{y}_i . \tag{49}$$

An inverse transform can now be applied to construct an estimate of $\hat{u}(x)$,

$$\hat{u}^*(x) = \sum_{i=0}^{\infty} \hat{u}_i^* \hat{\phi}_i(x). \tag{50}$$

With $\hat{u}^*(x)$ obtained from (50), the estimate $u^*(x)$ of the original object $u(x)$ is then obtained by adding the boundary-response term of (32), i.e.,

$$u^*(x) = \hat{u}^*(x) + u(x)\, u(-L) + b(x)\, u(L). \tag{51}$$

It is seen from (18) that the $\{w_i\}$ are zero mean random numbers. If $w(x)$ extends only over the interval $[-L, L]$ or fills the entire interval $[-2L, 2L]$ then they are also uncorrelated. In some cases the noise may be present only over the interval $[-L - \Delta, L + \Delta]$ where Δ is the half-width of the PSF. With the $\phi_i(x)$ as given in (9):

$$\begin{aligned} E\{w_i w_j\} &= \sigma_n^2 \int_{-L-\Delta}^{L+\Delta} \phi_i(x)\, \phi_j(x)\, dx \\ &= \sigma_n^2 + 2\sigma_n^2 \int_{L}^{L+\Delta} \phi_i^2(x)\, dx, \qquad && i = j, \\ &= 2\sigma_n^2 \int_{L}^{L+\Delta} \phi_i(x)\, \phi_j(x)\, dx, && i + j \text{ even}, \\ &= 0, && i + j \text{ odd}. \end{aligned}$$

In such cases the filter equations derived here become approximations to the true Wiener filter, which has to be obtained in terms of a set of doubly orthogonal complete functions, as explained previously.

Figures 2–5 display the results of this technique. Figure 2 is an image blurred by a known symmetric point spread function with no additive noise. Figure 3 is the restored image which is identical to the original unblurred image. Figure 4 is the blurred image with additive white Gaussian noise. The signal-to-noise ratio is about 6 db. Figure 5 is the restored image. Note that the noise has been removed but the image appears blurred. This is due to the loss of high-frequency components due to filtering.

Extensions and Conclusions

The extension of the present approach to the cases where the PSF is spatially invariant, but nonsymmetric (e.g., in the case of motion blur), is also possible. The derivation for this extension is given in the Appendix. Extensions to the problems where PSF is a random function [8] (e.g., in astronomical speckle patterns) also seem possible. In conclusion, it was shown that the Wiener filter solution of the image restoration problem over a finite interval requires a doubly orthogonal set of basis functions which is difficult to obtain. However, if this problem is formulated via a boundary response decomposition of the object function and the image observations, then a fast implementation of the solution follows. Similar results are also possible for discrete objects. The derivation there is more complex and requires a special kind of factorization of the PSF represented by a matrix [5].

APPENDIX: Fast Wiener Restoration for Nonsymmetric Finite PSF

If the PSF is nonsymmetric, a similar expansion is possible. In this case (44) becomes

$$h(x) = \frac{1}{2L} \sum_{i=0}^{\infty} \left(h_{ci} \cos \frac{i\pi}{2L} x + h_{si} \sin \frac{i\pi}{2L} x\right), \qquad -2L \leqslant x \leqslant 2L,$$

$$h_{ci} = \int_{-2L}^{2L} h(x) \cos \frac{i\pi}{2L} x \, dx,$$

$$h_{si} = \int_{-2L}^{2L} h(x) \sin \frac{i\pi}{2L} x \, dx. \tag{52}$$

Substituting into (43):

$$\begin{aligned} \hat{y}(x) &= \frac{1}{2L} \sum_{i=0}^{\infty} h_{ci} \int_{-L}^{L} \hat{u}(\beta) \cos \frac{i\pi}{2L} (x - \beta) \, d\beta \\ &\quad + \frac{1}{2L} \sum_{i=0}^{\infty} h_{si} \int_{-L}^{L} \hat{u}(\beta) \sin \frac{i\pi}{2L} (x - \beta) \, d\beta + w(x) \\ &= \frac{1}{2L^{1/2}} \sum_{i=0}^{\infty} \Big[(h_{ci}\hat{u}_{ci} - h_{si}\hat{u}_{si}) \cos \frac{i\pi}{2L} x \\ &\quad + (h_{ci}\hat{u}_{si} + h_{si}\hat{u}_{ci}) \sin \frac{i\pi}{2L} x\Big] + w(x), \qquad -2L \leqslant x \leqslant 2L, \end{aligned} \tag{53}$$

where

$$\hat{u}_{ci} = \left(\frac{1}{L}\right)^{1/2} \int_{-L}^{L} \hat{u}(\beta) \cos \frac{i\pi}{2L} \beta \, d\beta,$$

$$\hat{u}_{si} = \left(\frac{1}{L}\right)^{1/2} \int_{-L}^{L} \hat{u}(\beta) \sin \frac{i\pi}{2L} \beta \, d\beta.$$

Correlating the observed signal $\hat{y}(x)$ with harmonic sinusoids yields

$$\begin{aligned}
\hat{y}_{cj} &= \left(\frac{1}{L}\right)^{1/2} \int_{-2L}^{2L} \hat{y}(x) \cos \frac{j\pi}{2L} x \, dx = h_{cj}\hat{u}_{ij} - h_{sj}\hat{u}_{sj} + w_{cj} , \\
\hat{y}_{sj} &= \left(\frac{1}{L}\right)^{1/2} \int_{-2L}^{2L} \hat{y}(x) \sin \frac{j\pi}{2L} x \, dx = h_{cj}\hat{u}_{sj} + h_{sj}\hat{u}_{cj} + w_{sj} , \\
w_{cj} &= \left(\frac{1}{L}\right)^{1/2} \int_{-2L}^{2L} w(x) \cos \frac{j\pi}{2L} x \, dx, \\
w_{sj} &= \left(\frac{1}{L}\right)^{1/2} \int_{-2L}^{2L} w(x) \sin \frac{j\pi}{2L} x \, dx,
\end{aligned} \tag{54}$$

and finally ...,

$$\begin{aligned}
Z_{1j} &\triangleq \frac{h_{cj}\hat{y}_{cj} + h_{sj}\hat{y}_{sj}}{h_{cj}^2 + h_{sj}^2} = \hat{u}_{cj} + w_{1j} , \\
Z_{2j} &\triangleq \frac{-h_{sj}\hat{y}_{cj} + h_{cj}\hat{y}_{sj}}{h_{cj}^2 + h_{sj}^2} = \hat{u}_{sj} + w_{2j} , \\
w_{1j} &= \frac{h_{cj}w_{cj} + h_{sj}w_{sj}}{h_{cj}^2 + h_{sj}^2} , \quad w_{2j} = \frac{h_{cj}w_{sj} - h_{sj}w_{cj}}{h_{cj}^2 + h_{sj}^2} .
\end{aligned} \tag{55}$$

Only one of these values is needed for the eigenfunction expansion: the cosine term if j is odd or the sine term if j is even. The noise terms will be exactly uncorrelated for all j except when noise and observation intervals are different. (See Eq. (51).)

Once again both $\{Z_{1j}\}$ and $\{Z_{2j}\}$ are uncorrelated sequences.

$$\begin{aligned}
Z_j &= Z_{1j} , \qquad j = 1, 3, 5, \ldots, \\
&= Z_{2j} , \qquad j = 2, 4, 6, \ldots, \\
\hat{u}_j^* &= \frac{h_i \lambda_i}{h_i^2 \lambda_i + \sigma_n^2} Z_j .
\end{aligned} \tag{56}$$

Again the inverse transform gives $\hat{u}^*(x)$ as in (50) and $u^*(x)$ as in (51).

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